Smith **Math 290** **Fall 2009**

Exam 2

I affirm this work abides by the university's Academic Honesty Policy.

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Print Name, then Sign

Directions:

- **Only write on one side of each page.**
- **Use terminology correctly.**
- **Partial credit is awarded for correct approaches so justify your steps.**

Do both of the following

D.1. [5 points] Use correct notation to write an arbitrary **relation of linear dependence for the set of vectors** $S = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$.

D.2. [5 points] Use correct notation to write the ith entry in the **matrix-vector product** $A\mathbf{u}$ of the $m \times n$ matrix A having columns $\mathbf{A}_1, \mathbf{A}_2. \cdots, \mathbf{A}_n$ with the vector $\mathbf{u}$ of size n.

Do any two (2) of these "Computational" problems

C.1. [15 points] Find the column space of $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 4 & -14 \\ 3 & 10 & -2 \\ 3 & -1 & 20 \\ 6 & 9 & 18 \end{bmatrix}$ in two different ways. For both, write the column space as the span of a linearly independent set of vectors.

C.2. [15 points] The matrix $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ has the property that $A\overrightarrow{x} = 5\overrightarrow{x}$ for some vectors $\vec{x}$. Write the set of such vectors as the span of a linearly independent set.

C.3. [15 points] Consider the following vectors in $\mathbf{C}^4$.

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Find a vector $\overrightarrow{u}_4$ in $\mathbf{C}^4$ so that $\overrightarrow{u}_1$, $\overrightarrow{u}_2, \overrightarrow{u}_3, \overrightarrow{u}_4$ form an orthonormal set.

Useful Information:

1. The set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an **orthonormal** set.
2. The Gram-Schmidt formula is

$$\vec{u}_i = \vec{v}_i - \left(\frac{<\vec{v}_i, \vec{u}_1>}{<\vec{u}_1, \vec{u}_1>}\right)\vec{u}_1 - \cdots - \left(\frac{<\vec{v}_i, \vec{u}_{i-1}>}{<\vec{u}_{i-1}, \vec{u}_{i-1}>}\right)\vec{u}_{i-1}$$

Do any two (2) of these "In Class, Text, or Homework" problems

M.1. [15 points] Theorem MIT (Matrix Inverse of a Transpose) in our book says that if A is an invertible matrix, then so is A^t and $\left(A^t\right)^{-1} = \left(A^{-1}\right)^t$. Prove this theorem.

M.2. [15 points] Prove that if A in an $m \times n$ matrix and B is $n \times p$ then the column space of AB is contained in the column space of A. That is, prove $C(AB) \subseteq C(A)$.

M.3. [15 points] Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors from $\mathbf{C}^m$. Prove the following set equality. $\langle\{\mathbf{v}_1, \mathbf{v}_2\}\rangle = \langle\{\mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{v}_1 - \mathbf{v}_2\}\rangle$.

Do any two (2) of these "Other" problems

T.1. [15 points] Is it possible to have an invertible 3×3 matrix A with the property that $A^2 = O_3$? Why or why not? (Here O_3 denotes the 3×3 zero matrix.)

T.2. [15 points] Suppose $A_{n\times m}$ and $B_{m\times n}$ are matrices such that $AB = I_n$. Let $\overrightarrow{b}$ be a particular vector in $\mathbf{C}^n$. Show that the system of equations $A\overrightarrow{x} = \overrightarrow{b}$ must be consistent.

T.3. [15 points] Our author (Beezer) proved in one of the textbook exercises that if If $\vec{u}_1$ and $\vec{u}_2$ are both in $\langle S\rangle$, the span of S, then so is the sum $\vec{u}_1 + \vec{u}_2$. Use Beezer's result and the Principle of Mathematical Induction to prove that if $\vec{u}_1, \vec{u}_2, \vec{u}_3, \cdots, \vec{u}_n$ are all in $\langle S\rangle$ then so is the sum $\vec{u}_1 + \vec{u}_2 + \vec{u}_3 + \cdots + \vec{u}_n$.

T.4. [15 points] Suppose that $\vec{a}$ and $\vec{b}$ are solution vectors to the non-homogeneous linear system of equations $A\vec{x} = \vec{c}$. Prove that $\vec{a} - \vec{b}$ is a solution vector to the homogeneous system $A\vec{x} = \vec{0}$.